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Towards an exhaustive classification of the star–triangle relation: II

Jean-Marie Maillard† and Thomas Garel‡

† DPhG/PSRM, CEN Saclay, Orme des Merisiers, 91191 Gif-sur-Yvette Cédex, France
‡ Laboratoire de Physique des Solides§, Bat. 510, Université Paris-Sud, Centre d'Orsay, 91405 Orsay Cédex, France

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Abstract. We continue our study of the star–triangle relation for two-component spin models, with the help of the inversion relation. The conclusion is that the most general interaction-round-a-face models (16 non-zero parameters) that satisfy the star–triangle relations are mainly Baxter's and Fan and Wu's models. In this way we recover Krichever's result for the classification of the star–triangle relations for vertex models. We sketch a generalisation of this approach to models with excluded configurations, to staggered models, and to three-dimensional models (tetrahedron relations).

1. Introduction

The motivations for the present study and the definitions of variables have been thoroughly exposed in the preceding paper (Maillard and Garel 1984, to be referred to as (I)). We will focus here on the $\rho = \lambda\mu$ case, with all parameters different from zero, for which some results have been presented in (I). To go deeper into this relevant case, we will make use of the inversion relation (IR) (Stroganov 1979) whenever it exists; more precisely we will use the stability of the star–triangle relation (STR) under the IR.

There remain the singular cases for which the inversion relation does not exist. In another context, there also remain the degenerate cases where some Boltzmann weights, associated with special spin configurations, vanish (model with exclusions).

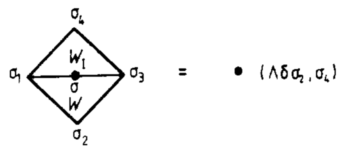
Section 2 will deal with the IR and we will show that the algebraic invariants, obtained from the commutation of transfer matrices of small size, are stable under the IR.

In § 3 we will study the case $\rho = \lambda\mu$ (non-zero parameters) and show, with the help of the IR, that the non-singular cases (for which an IR does exist) are mainly Baxter's and Fan and Wu's models (Baxter 1972, Fan and Wu 1970). Possible extensions of the present method will be sketched in § 4 for (i) singular models for which an IR does not exist and models with excluded configurations such as the hard hexagon model, (ii) some staggered models, (iii) three-dimensional models (tetrahedron relation).

§ Laboratoire associé au CNRS.

2. Inversion relation and stability of algebraic invariants

The inversion relation (IR) (Stroganov 1979) has been used by many authors as a short cut to calculate the partition function (e.g. Baxter 1980a, Schultz 1981). Let us define the IR in the case of the interaction-round-a-face model (IRF): the IR means that the partition functions of the two graphs below are equal:



Here W_I stands for the inverse Boltzmann weight of W; σ_1 and σ_3 are fixed and one sums over the spin σ; Λ is some known factor. This relation analytically means that for any fixed configuration (σ_1, σ_3) one has

$$\sum_{\sigma} W(\sigma_1, \sigma_2, \sigma_3, \sigma)\, W_I(\sigma_1, \sigma, \sigma_3, \sigma_4) = \Lambda\delta\sigma_2, \sigma_4. \tag{1}$$

In essentially all the known soluble two-dimensional models, one may check that the STR and the IR occur simultaneously. Moreover, the STR has a nice stability property with respect to the IR.

Acting simultaneously on the $(\sigma_1, \sigma_6, \sigma_5)$ and $(\sigma_2, \sigma_3, \sigma_4)$ sides of the two hexagons of the STR (figure 1) by the inverse Boltzmann weight W_I, and using the definition of

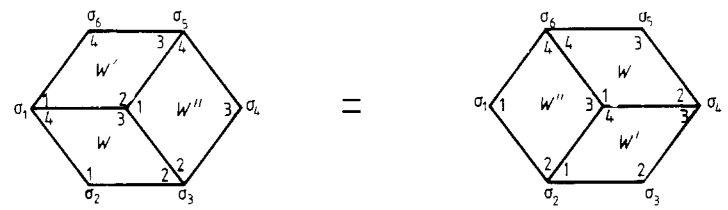


Figure 1. The star–triangle relation.

the IR, one gets a new STR (figure 2). Of course there exist similar STR involving W_I and W_I''. This new STR (figure 2) implies as well a new transfer matrix commutation

$$[T_N(W), T_N(W_I')] = 0. \tag{2}$$

From $[T_N(W), T_N(W_I')] = 0$ (see equation (2) of paper (I)) we have obtained a set of algebraic equations for small size N $(N = 1, 2, \ldots)$: $\varphi_\alpha(W) = \varphi_\alpha(W')$. Similarly we get from (2)

$$\varphi_\alpha(W) = \varphi_\alpha(W_I')$$

and therefore

$$\varphi_\alpha(W') = \varphi_\alpha(W_I'). \tag{3}$$

The algebraic invariants φ_α have to be stable under the IR.

More precisely, the algebraic invariants φ_α coming from the horizontal transfer matrix commutation have to be stable under the IR denoted by I (see appendix);

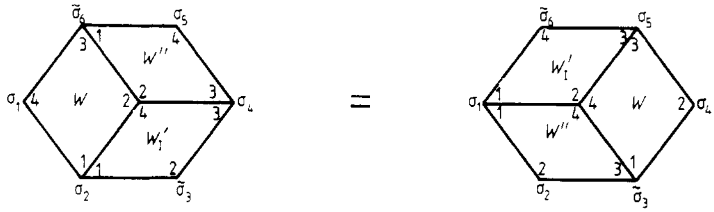


Figure 2. The I-transform of the star-triangle relation.

clearly the φ_α's are also stable under the inverse (in the group sense) of transformation I that we denote by $J = I^{-1}$. In a similar way the algebraic invariants $\tilde{\varphi}_\alpha$ associated to the vertical transfer matrix commutation (see paper (I)) have to be stable under their corresponding IR denoted by K and its inverse $L = K^{-1}$; both are also made explicit in the appendix. Obviously the above considerations are appropriate for the non-singular cases where an IR does exist.

3. The non-singular $\rho = \lambda\mu$ case

3.1. Preliminaries

As shown in paper (I), we can extract from the case $N = 1$ and 2 at least four algebraic invariants (horizontal transfer matrix)

$$(a-p)/d, \qquad bc/ei, \qquad no/hl, \qquad (d/m)^2.$$

Let us use the stability property under the inverse I

$$d/m = d_I/m_I = (k/f)(bh - fd)/(io - km) \tag{4}$$

where $d_I, m_I, \ldots$ denote the I-transforms of $d, m, \ldots$. We also get

$$bc/ei = b_I c_I/e_I i_I = (oc/eh)(bh - fd)/(io - km) \tag{5}$$

and

$$hl/no = h_I l_I/n_I o_I = (li/bn)(bh - fd)/(io - km). \tag{6}$$

From (4), (5), (6) we have

$$df/km = bh/io, \qquad kd/fm = cl/en, \tag{7), (8}$$

which in turn imply

$$(d/m)^2 = (bc/ei)(hl/no) \tag{9}$$

i.e. $\rho = \lambda\mu$ and

$$(f/k)^2 = (eb/ic)(nh/ol). \tag{10}$$

We therefore conclude that the mere existence of an IR is sufficient to impose the condition $\rho = \lambda\mu$ (equation (1)).

No new invariants are generated by the inverse $J = I^{-1}$. We would get the same equations (7) and (8) by considering the stability of the $\tilde{\varphi}_\alpha$'s under the inverse K (and L).

Equations (4)–(10) thus behave as a closed set under the IR; to find new invariants φ_α requires a different approach.

Let us now consider the remaining invariants, for $N=1$ for the horizontal $((a-p)/d)$ and vertical $((a-p)/f)$ transfer matrices; these invariants are stable respectively under I and K. If $a-p\neq 0$ this yields

$$\frac{a-p}{d}=\frac{g/\Delta_1-j/\Delta_4}{-k/\Delta_2}, \tag{11}$$

$$\frac{a-p}{f}=\frac{g/\Delta_1-j/\Delta_4}{-m/\Delta_2}, \tag{12}$$

where the Δ_i's are defined in the appendix; from (11) and (12) we get $kf=dm$. Successive I transforms of (11) will lead to an infinite number of constraints (as opposed to the closed set (4)–(10)). For this reason, it seems unlikely to find any non-trivial solution satisfying these constraints. From now on, we will therefore assume that

$$a=p. \tag{13}$$

Transforming (13) by I and J, we have

$$g/j=\Delta_1/\Delta_4=\delta_4/\delta_1=(\Delta_1/\Delta_4)_J \tag{14}$$

and therefore

$$(\Delta_1/\Delta_4)_I=(\Delta_1/\Delta_4)_J, \tag{15}$$

$$(\delta_1/\delta_4)_I=(\delta_1/\delta_4)_J. \tag{16}$$

Writing (15) explicitly, we obtain

$$\frac{gp/\Delta_1\Delta_4-ho/\Delta_2\Delta_3}{aj/\Delta_1\Delta_4-bi/\Delta_2\Delta_3}=\frac{gp-ho}{aj-bi} \tag{15'}$$

and this gives two classes of solutions

$$(\alpha)\,ajho=bigp \qquad \text{or} \qquad (\beta)\,\Delta_1\Delta_4=\Delta_2\Delta_3.$$

A similar result can be obtained from (16), namely

$$(\alpha')\,agln=jpec \qquad \text{and} \qquad (\beta')\,\delta_1\delta_4=\delta_2\delta_3.$$

(Note that (α') and (β') are transforms of (α) and (β) under the IR.)

Before studying these two cases, let us remark that (15) and (16) imply that the ratio Δ_1/Δ_4 (or δ_1/δ_4) is 'special'. This 'special' role can also be appreciated in the following way: if one considers particular spin configurations of σ_2, σ_3, σ_5, σ_6 (figure 1) and makes use of the stability under the IR, it is possible to get, among many, the equations

$$\frac{\Delta_1''}{\Delta_4''}=\frac{\Delta_4}{\Delta_1}\frac{\Delta_1'}{\Delta_4'}=\frac{\delta_1}{\delta_4}\frac{\delta_4'}{\delta_1'}=\left(\frac{\Delta_4\Delta_1'}{\Delta_1\Delta_4'}\right)_I, \tag{17}$$

$$\frac{\Delta_1\delta_1}{\Delta_4\delta_4}=\frac{\Delta_1'\delta_1'}{\Delta_4'\delta_4'}=\frac{\Delta_1''\delta_1''}{\Delta_4''\delta_4''}. \tag{18}$$

3.2. Baxter-like solution

We wish to study here the case (α) (or (α')) $ajho = bigp$ and

$$agln = jpec \qquad \text{(with } a = p\text{)}.$$

Let us now use the gauge transformations D_1 and D_2 introduced in paper (I). Owing to their overlap $D_1 \cap D_2$, which depends on one parameter, we can impose $g = j$ which, due to (14), implies $\Delta_1 = \Delta_4$,

$$ln = ec \qquad \text{and} \qquad bi = ho. \tag{19),(20}$$

Similarly making use of the non-overlapping parts ($D_1 - D_2$ and $D_2 - D_1$), we can fix

$$d = m \qquad \text{and} \qquad f = k. \tag{21),(22}$$

Recalling (7) and (8) we therefore have

$$bh = io \qquad \text{and} \qquad cl = en \tag{23),(24}$$

which in turn lead to

$$b = \varepsilon o, \qquad i = \varepsilon h, \tag{25),(26}$$

$$c = \varepsilon' n, \qquad e = \varepsilon' l. \tag{27),(28}$$

The stability of (25) under the IR implies $\varepsilon = \varepsilon'$. These simplifications (reduction from 16 to 8 parameters) allow us to thoroughly investigate the algebraic invariants $\varphi_\alpha(W)$ for $N = 3$; in this case the invariants quoted in paper (I) become trivial, but the small number of parameters enables us to get new algebraic invariants such as

$$\frac{a^3 \pm d^3 - (af^2 + gec + \varepsilon gbh) \mp (g^2 d + fbh + \varepsilon fec)}{bc(a \pm d)}.$$

If $\varepsilon = 1$, that is, restricting ourselves to positive Boltzmann weight ($a = p$, $b = o$, $c = n$, $e = l$, $i = h$), we could continue the analysis for larger N and get in this way the symmetric 8-vertex (Baxter 1972), namely $c = e$, $d = f$, $a = g$, $b = h$, and Fan and Wu's model (see below).

At this point it is simpler to observe that the spin reverse property holds

$$W(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = W(-\sigma_1, -\sigma_2, -\sigma_3, -\sigma_4).$$

Through the Kadanoff–Wegner mapping (Kadanoff and Wegner 1971), this spin model can be viewed as an asymmetric 8-vertex model (8 homogeneous parameters).

We can therefore rely on Krichever's analysis (1981) (see also Sogo *et al* 1982) to conclude that the most general models of this case ($\varepsilon = 1$), that satisfies the STR, are the symmetric 8-vertex model (Baxter 1972) and Fan and Wu's model that we now consider.

4. Fan–Wu-like solution

Let us now study solution (β):

$$\Delta_1\Delta_4 = \Delta_2\Delta_3 \qquad \text{and} \qquad \delta_1\delta_4 = \delta_2\delta_3.$$

Using the same line of argument as for solution (α), we can restrict ourselves to an 8-parameter model ($d = m$, $f = k$, $b = \varepsilon o$, $c = \varepsilon n$, $e = \varepsilon l$, $i = \varepsilon h$, $\Delta_2 = \Delta_3$, $\delta_2 = \delta_3$, $g = j$,

$\Delta_1 = \Delta_4$) obeying

$$(ag - ec)^2 = (bh - fd)^2, \tag{29}$$

$$(ag - \varepsilon bh)^2 = (\varepsilon ec - fd)^2. \tag{30}$$

The study of non-singular solutions of (29) and (30) splits into three cases

(β.1) $ag = \varepsilon bh$ and $ec = \varepsilon fd$,

(β.2) $ag = \varepsilon fd$ and $ec = \varepsilon bh$,

(β.3) $ag - ec = \varepsilon(bh - fd)$.

If we restrict ourselves to positive Boltzmann weights ($\varepsilon = 1$), solution (β.3) is the Fan–Wu free fermion condition (Fan and Wu 1970). The model is not in general included in the Baxter model (existence of non-zero fields). The cases (β.1), (β.2) need further study.

The section can be summarised as follows: the classification of the spin STR at least for non-degenerate cases yields solutions very similar to the vertex STR classification; Baxter's and Fan and Wu's models are recovered. The similarity breaks down for degenerate models (hard hexagon).

4. Extension of the method

4.1. Singular and degenerate cases

(a) *Singular cases*
The study of models which do not possess an IR is not much advanced: some of these models are zero- or one-dimensional in disguise, but others are quite non-trivial (isotropic two-dimensional Ising model with a magnetic field).
(b) *Degenerate cases*
When one or more of the parameters are zero, the methods of paper (I) and of this paper still apply, but the results differ. (The commutation of transfer matrices may or may not give the same algebraic invariants). An exhaustive classification of these cases appears to be an extremely tedious task (numerous possible ramifications).

A noteworthy example is the hard hexagon model, which corresponds to $a = b = c = d = e = f = i = k = m = 0$ (Baxter 1980b).

The method gives for $N = 2, 3, 4$ respectively three invariants $\varphi_1, \varphi_2, \varphi_3$ which are not independent since one can show that $(\varphi_2)_J = (\varphi_2)_I = \varphi_3 - \varphi_1$, which in turn implies (§ 2) $\varphi_2 = \varphi_3 - \varphi_1$. This relation is actually satisfied for the hard hexagon model: see for instance equation (23) of Baxter (1980b). The hard hexagon model is definitively different from Baxter's or Fan and Wu's (e.g. different critical exponent δ).

4.2. Staggered models

Let us consider a staggered model possessing a spin reverse property with Boltzmann weights W_1, W_2 (see figure 3).

This model is equivalent to a staggered asymmetric 8-vertex model and contains important subcases such as the Ashkin–Teller model and the non-critical Potts model (staggered 6-vertex model). The calculation for $N = 1$ (noting the doubling of the cell

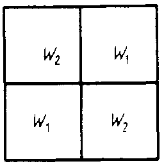


Figure 3. Elementary cell for a staggered model.

in both directions) gives the following invariants:

$$\frac{a_1a_2+d_1d_2-f_1f_2-g_1g_2}{c_1b_2+b_1c_2}, \qquad \frac{e_1h_2+h_1e_2}{b_1c_2+c_1b_2},$$

$$\frac{e_1h_2-h_1e_2}{b_1c_2-c_1b_2}, \qquad \frac{a_1a_2+f_1f_2-d_1d_2-g_1g_2}{e_1h_2-h_1e_2}.$$

5. Three-dimensional generalisations

The three-dimensional generalisation of the STR is called the tetrahedron relation (TeR), and it also implies the commutation of transfer matrices of finite size (N, M), provided one takes periodic boundary conditions in two directions: $[T_{N,M}(W), T_{N,M}(W')]=0$.

These commutation relations yield algebraic invariants $\varphi_\alpha(W)=\varphi_\alpha(W')$; for instance, in the case of Zamolodchikov's solution of the tetrahedron relation (Zamolodchikov 1981) we find for $N=M=2$ the invariant†

$$\varphi(W)=\frac{[(P_0-Q_0)^4-(P_0+Q_0)^4+(P_1-Q_1)^4-(P_1+Q_1)^4+(P_2+Q_2)^4-(P_2-Q_2)^4+(P_3-Q_3)^4-(P_3+Q_3)^4]}{R_0R_1R_2R_3}$$

(where we have used Baxter's notation (1983)). The method presented in § 2 can be used to generate new invariants with the help of the three-dimensional IR. Such a task appears to be a tedious but systematic one, and may lead to a classification of the TeR.

6. Conclusion

In this paper and in paper (I), we have tried to classify the STR for two-component spin models. In the non-degenerate cases, our results are as follows: if there is an inversion relation (non-singular case), necessarily $\rho=\lambda\mu$ and the most general models that satisfy the STR are mainly Baxter's and Fan and Wu's.

If there is no IR, we can only conclude if $\rho\neq\lambda\mu$ and if the algebraic invariants $\varphi_\alpha(W)$ are independent: in this case the only solution is the trivial solution $W=\text{constant}\times W'$. The degenerate cases can be, in principle, studied in the same way.

From a general point of view, this problem of exhaustive classification is similar to the problem of classification one encounters in group theory.

† With Baxter's angles (Baxter 1983) $\varphi(W)$ is equal to $\cos\phi_2$.

It is not easy to use perturbation theory for this purpose as is clear from the very hairy structure of models having a STR: Fan and Wu's solution is not completely included in Baxter's, neither is the asymmetric 6-vertex solution. The hard hexagon model does not seem to belong at all to this set (different critical exponents). It seems to us that the method and results presented in these two papers are well adapted to describe such a structure.

Appendix

We define here the four (inverse) transformations I, J, K, L by the following matrix identities:

$$\begin{pmatrix} a_I & b_I \\ i_I & j_I \end{pmatrix}\begin{pmatrix} a & c \\ e & g \end{pmatrix}=\begin{pmatrix} e_I & f_I \\ m_I & n_I \end{pmatrix}\begin{pmatrix} b & d \\ f & h \end{pmatrix}$$

$$=\begin{pmatrix} c_I & d_I \\ k_I & l_I \end{pmatrix}\begin{pmatrix} i & k \\ m & o \end{pmatrix}=\begin{pmatrix} g_I & h_I \\ o_I & p_I \end{pmatrix}\begin{pmatrix} j & l \\ n & p \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{A1}$$

$$\begin{pmatrix} a_J & c_J \\ e_J & g_J \end{pmatrix}\begin{pmatrix} a & b \\ i & j \end{pmatrix}=\begin{pmatrix} b_J & d_J \\ f_J & h_J \end{pmatrix}\begin{pmatrix} e & f \\ m & n \end{pmatrix}$$

$$=\begin{pmatrix} i_J & k_J \\ m_J & o_J \end{pmatrix}\begin{pmatrix} c & d \\ k & l \end{pmatrix}=\begin{pmatrix} j_J & l_J \\ n_J & p_J \end{pmatrix}\begin{pmatrix} g & h \\ o & p \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{A2}$$

$$\begin{pmatrix} a_K & i_K \\ b_K & j_K \end{pmatrix}\begin{pmatrix} a & c \\ e & g \end{pmatrix}=\begin{pmatrix} e_K & m_K \\ f_K & n_K \end{pmatrix}\begin{pmatrix} i & k \\ m & o \end{pmatrix}$$

$$=\begin{pmatrix} e_K & k_K \\ d_K & l_K \end{pmatrix}\begin{pmatrix} b & d \\ f & g \end{pmatrix}=\begin{pmatrix} g_K & o_K \\ h_K & p_K \end{pmatrix}\begin{pmatrix} j & l \\ n & p \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{A3}$$

$$\begin{pmatrix} a_L & c_L \\ e_L & g_L \end{pmatrix}\begin{pmatrix} a & i \\ b & j \end{pmatrix}=\begin{pmatrix} i_L & k_L \\ m_L & o_L \end{pmatrix}\begin{pmatrix} e & m \\ f & n \end{pmatrix}$$

$$=\begin{pmatrix} b_L & d_L \\ f_L & h_L \end{pmatrix}\begin{pmatrix} c & k \\ d & l \end{pmatrix}=\begin{pmatrix} j_L & l_L \\ n_L & p_L \end{pmatrix}\begin{pmatrix} g & o \\ h & p \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{A4}$$

Moreover, we set (see (11) and (12))

$$\begin{aligned} \Delta_1 &= ag-ec, & \delta_1 &= aj-bi, \\ \Delta_2 &= io-km, & \delta_2 &= cl-kd, \\ \Delta_3 &= bh-fd, & \delta_3 &= en-fm, \\ \Delta_4 &= jp-ln, & \delta_4 &= gp-ho. \end{aligned}$$

In degenerate cases such as the hard hexagon model one has to adopt the following convention: the inverse of $\left(\begin{smallmatrix} 0 & 0 \\ 0 & x \end{smallmatrix}\right)$ is $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1/x \end{smallmatrix}\right)$ and similarly $\left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right)$ has the inverse $\left(\begin{smallmatrix} 1/x & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

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